Gap solitons in damped and parametrically driven nonlinear diatomic lattices

Guoxiang Huang
China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, China;
International Center for Materials Physics, Academia Sinica, Shenyang 110015, China;
*Department of Physics, East China Normal University, Shanghai 200062, China**
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Considerable theoretical progress has been made in the study of gap solitons in nonlinear periodic media and anharmonic diatomic lattices. In this paper we investigate the soliton excitations in a damped and parametrically driven nonlinear diatomic lattice. An experiment for observing gap solitons is suggested.

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The formation and dynamics of localized structures in nonlinear systems outside of equilibrium are currently the subjects of intense theoretical and experimental efforts [1]. Examples include the parametrically excited surface waves, elastic media, electroconvection in nematic liquid crystals, and linear arrays of vortices, etc. In a recent experiment, Denardo *et al.* found that a damped and parametrically driven nonlinear coupled pendulum lattice can have a wealth of nonpropagating self-localized structures. Steady-state patterns, such as the standing solitons, kinks, and domain walls were observed [2]. Several analytical and simulation studies of these localized states have been proposed [3–6]. Nevertheless, there are some problems remaining to be solved in theory and experiment [3].

On the other hand, the so-called intrinsic localized modes in anharmonic atomic chains have been receiving much attention due to Sievers and Takeno [7–10]. Recently the study has turned to the *gap solitons* in nonlinear diatomic lattices [11–14]. The concept of the gap soliton was first introduced by Chen and Mills when they studied the nonlinear optical response of superlattices [15]. However, to the author's knowledge, there is no experimental report on the observation of gap solitons up to now. In this paper we investigate the localized structures in a damped and parametrically driven anharmonic "diatomic" lattice. An experiment for observing gap solitons is suggested.

We consider a coupled pendulum lattice, which is basically the same as the one used by Denardo *et al.* for observing solitons, kinks, and domain walls in a "monatomic" pendulum lattice [2,6], but the bottom of the jth pendulum is a small ball with mass m_j. Then for the jth pendulum we have the following equation of motion:

$$m_j\frac{d^2\theta_j}{dt^2}-K_2(\theta_{j+1}+\theta_{j-1}-2\theta_j)+\beta_j\frac{d\theta_j}{dt}+m_j[\omega_0^2+\eta\cos(2\omega_e t)]\theta_j-\alpha m_j\theta_j^3=0\ , \quad (1)$$

where the dots represent time differentiation, θ_j is the angle of deflection of the jth pendulum, K_2 is a measure of the coupling strength, ω_0 is the linear frequency of an uncoupled pendulum, η is the drive amplitude, $2\omega_e$ is the drive frequency, β_j is the damping parameter, and α is the nonlinear coefficient. For the diatomic lattice we can assume that $\theta_{2k}=v_n$, $m_{2k}=m$, $\beta_{2k}=\beta_1$ for $j=2k$ (even particles), and $\theta_{2k+1}=w_n$, $m_{2k+1}=M$, $\beta_{2k+1}=\beta_2$ for $j=2k+1$ (odd particles). n is the index of the nth unit. Then (1) can be split into two equations:

$$\frac{d^2v_n}{dt^2}-I_2(w_n+w_{n-1}-2v_n)+\gamma_1\frac{dv_n}{dt}+[\omega_0^2+\eta\cos(2\omega_e t)]v_n-\alpha v_n^3=0\ , \quad (2)$$

$$\frac{d^2w_n}{dt^2}-J_2(v_{n+1}+v_n-2w_n)+\gamma_2\frac{dw_n}{dt}+[\omega_0^2+\eta\cos(2\omega_e t)]w_n-\alpha w_n^3=0\ , \quad (3)$$

where $I_2=K_2/m$, $J_2=K_2/M$, $\gamma_1=\beta_1/m$, and $\gamma_2=\beta_2/M$. The linear dispersion relation when neglecting the damping and drive is

$$\omega^2=\omega_0^2+I_2+J_2\pm[(I_2+J_2)^2-4I_2J_2\sin^2(qa)]^{1/2}\ , \quad (4)$$

where a is the lattice constant and ω and q are the frequency and wave number of the linear wave, respectively. The minus corresponds to the low-frequency mode (lower or "acoustic" branch), and the plus corresponds to the high-frequency mode (upper or "optical" branch). The modes are separated by the gap $\Delta\omega=\omega_2-\omega_1>0$, where $\omega_1^2=\omega_0^2+2J_2$ and $\omega_2^2=\omega_0^2+2I_2$. The most interesting region of the linear spectrum for the diatomic lattice is the vicinity of the maximum value of the wave number q_B [$q_B=\pi/(2a)$ is the Brillouin-zone edge], where two branches with opposite signs of the dispersion are separated by the gap. Near $q=\pi/(2a)$, either heavy (lower branch) or light (upper branch) atoms vibrate almost with opposite phases and a long-wavelength approximation may be used for these opposite vibrations [4,5,12,13]. So in the vicinity of the point $q=\pi/(2a)$ we can make the ansatz

*Mailing address.

$$v_n=(-1)^n[V(\xi_n,\tau)\exp(i\omega t)+V^*(\xi_n,\tau)\exp(-i\omega t)]\ , \quad (5)$$

$$w_n=(-1)^n[W(\xi_n,\tau)\exp(i\omega t)+W^*(\xi_n,\tau)\exp(-i\omega t)]\ , \quad (6)$$

where $\xi_n=\epsilon^2 2na$ and $\tau=\epsilon^2 t$ are slow variables, ϵ is a small (finite) parameter, and ω is a frequency parameter which has not been specified in this stage. For small vibration we can assume that $[V(\xi_n,\tau), W(\xi_n,\tau)]=\epsilon[\tilde{V}(\xi_n,\tau),\tilde{W}(\xi_n,\tau)]$ and $(\eta,\gamma_1,\gamma_2,\omega_e-\omega)=\epsilon^2(\eta_1,\gamma_{10},\gamma_{20},\Delta\omega)$ with $\tilde{V}$, $\tilde{W}$, η_1, γ_{10}, γ_{20}, and $\Delta\omega$ being of $O(1)$ order. Substituting (5) and (6) into (2) and (3), retaining to $O(\epsilon^3)$ and making the rotating-wave approximation, we can obtain

$$2i\omega\frac{\partial V}{\partial t}+(\omega_2^2-\omega^2+i\omega\gamma_1)V-2aI_2\frac{\partial W}{\partial x}+\tfrac{1}{2}\eta V^*\exp[2i(\omega_e-\omega)t]-3\alpha|V|^2V=0\ , \quad (7)$$

$$2i\omega\frac{\partial W}{\partial t}+(\omega_1^2-\omega^2+i\omega\gamma_2)W+2aJ_2\frac{\partial V}{\partial x}+\tfrac{1}{2}\eta W^*\exp[2i(\omega_e-\omega)t]-3\alpha|W|^2W=0\ , \quad (8)$$

with $x=2na$, when returning to the original variables. The method of derivation for obtaining (7) and (8) is basically the same as the one used in Ref. [5]. To simplify (7) and (8) we make the following transformation [5]:

$$[V(x,t),W(x,t)]=[A(x,t),B(x,t)]\exp[i(\omega_e-\omega)t]\ . \quad (9)$$

Then we have

$$i\omega_e\frac{\partial A}{\partial t}+\tfrac{1}{2}(\omega_2^2-\omega_e^2+i\omega_e\gamma_1)A-aI_2\frac{\partial B}{\partial x}+\tfrac{1}{4}\eta A^*-\frac{3\alpha}{2}|V|^2V=0\ , \quad (10)$$

$$i\omega_e\frac{\partial B}{\partial t}+\tfrac{1}{2}(\omega_1^2-\omega_e^2+i\omega_e\gamma_2)B+aJ_2\frac{\partial V}{\partial x}+\tfrac{1}{4}\eta B^*-\frac{3\alpha}{2}|W|^2W=0\ . \quad (11)$$

In writing down (10) and (11) we have used the relation $2\omega(\omega_e-\omega)=2\omega(\omega_e^2-\omega^2)/(\omega_e+\omega)\simeq\omega_e^2-\omega^2$. (10) and (11) are two coupled nonlinear differential equations. To solve them, in general, is very difficult. But for a stationary solution we can assume that $\partial/\partial t=0$ and let $[A(x,t), B(x,t)]=[f_1(x), f_2(x)]\exp(i\phi)$, where f_j $(j=1,2)$ are real functions and ϕ is a real number to be determined later. Under the condition

$$\gamma_1=\gamma_2\ , \quad (12)$$

(10) and (11) transform into

$$\frac{df_1}{dx}=-\Delta_1 f_2+\sigma_1 f_2^3\ , \quad (13)$$

$$\frac{df_2}{dx}=\Delta_2 f_1-\sigma_2 f_1^3\ , \quad (14)$$

where $\Delta_1=[\omega_1^2-\omega_e^2+\frac{1}{2}\eta\cos(2\phi)]/(2aJ_2)$, $\Delta_2=[\omega_2^2-\omega_e^2+\frac{1}{2}\eta\cos(2\phi)]/(2aI_2)$, $\sigma_1=3\alpha/(2aJ_2)$, $\sigma_2=3\alpha/(2aI_2)$, and $\cos(2\phi)=[1-4\omega_e^2\gamma_1^2/\eta^2]^{1/2}$. The physical requirement for (12) will be discussed later.

It is not difficult to get a conservative quantity of (13) and (14):

$$G=2\Delta_2 f_1^2+2\Delta_1 f_2^2-\sigma_2 f_1^4-\sigma_1 f_2^4\ . \quad (15)$$

Thus (13) and (14) describe the dynamics of a Hamiltonian system with one degree of freedom. This point also can be shown by defining $f_1=p$, $f_2=q$, and $H(p,q)=G/4$. The Hamilton equation $dq/dx=\partial H/\partial p$ and $dp/dx=-\partial H/\partial q$ yield Eqs. (13) and (14). Here, x plays a role of "time." The solution can be obtained by introducing an auxiliary function $g=f_1/f_2$, which satisfies

$$\left[\frac{dg}{dx}\right]^2=(\Delta_1+\Delta_2 g^2)-G(\sigma_1+\sigma_2 g^4)\ . \quad (16)$$

Then f_1 and f_2 may be found by the relation

$$f_2^2=\frac{1}{\sigma_1+\sigma_2 g^4}\{(\Delta_1+\Delta_2 g^2)\pm[(\Delta_1+\Delta_2 g^2)-G(\sigma_1+\sigma_2 g^4)]^{1/2}\}\ , \quad (17)$$

$$f_1=gf_2\ . \quad (18)$$

It is helpful to use the method of qualitative analysis of dynamical systems and consider possible solutions of the system, (13) and (14), in the phase plane (f_1,f_2). Attention should be paid to separatrixes, which correspond to the soliton or kink solution of different kinds. The phase portrait of the system depends on the signs and on the values of the parameters Δ_1 and Δ_2, which change with increasing ω_e. *In experiment, ω_e, the external drive frequency, is an adjustable parameter.* As ω_e increases, a number of subsequent bifurcations (here antibifurcations) in the phase plane take place.

(i) For $\omega_e<[\omega_1^2+(1/2)\eta\cos(2\phi)]^{1/2}$, both Δ_1 and Δ_2 are positive and there are nine fixed points in the phase plane. It is easy to show that the fixed points (0,0) and $[\pm(\Delta_2/\sigma_2)^{1/2}, \pm(\Delta_1/\sigma_1)^{1/2}]$ are centers, and $[0,\pm(\Delta_1/\sigma_1)^{1/2}]$ and $[\pm(\Delta_2/\sigma_2)^{1/2}, 0]$ are saddle points. The phase portraits in this case have been shown in Fig. 1(a). There are two different types of separatrix curves. One is an *ABC* type (heteroclinic) orbit and the other one is an *EFE* type (homoclinic) orbit. The others can be obtained by symmetry. The field distribution corresponding to separatrixes *ABC* and *EFE* have been depicted in Figs. 1(b) and 1(c). We can obtain exact and explicit solutions corresponding to these separatrixes.

(a) ABC type. In this case $G=\Delta_1^2/\sigma_1$, and the boundary condition for g is $|g|=\infty$ for $x=\pm\infty$ and $|g|=\infty$ for $x=0$. Equation (16) may be easily integrated to give the solution

$$g(x)=\pm\frac{\alpha}{\beta}\frac{1}{\sinh(\beta x)}\ , \quad (19)$$

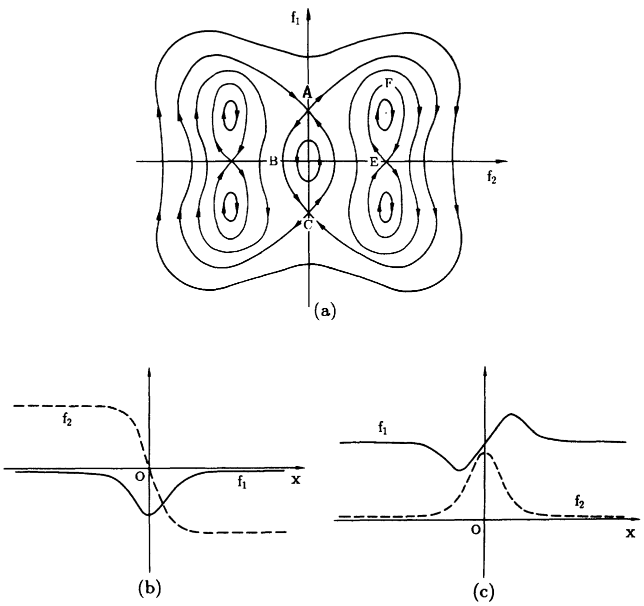

FIG. 1. (a) The phase portrait for $\omega_e < [\omega_1^2+\frac{1}{2}\eta\cos(2\phi)]$; (b) the field distribution corresponding to the heteroclinic orbit *ABC*; (c) the field distribution corresponding to the homoclinic orbit *EFE*.

$$f_2^2=\frac{1}{\sigma_1+\sigma_2 g^4}\left\{\Delta_1+\Delta_2 g^2\pm\alpha|g|\left[g^2+\left[\frac{\beta}{\alpha}\right]^2\right]^{1/2}\right\}, \tag{20}$$

with $f_1=gf_2$. Here $\alpha=[(\sigma_1\Delta_2^2-\sigma_2\Delta_1^2)/\sigma_1]^{1/2}$ and $\beta=(2\Delta_1\Delta_2)^{1/2}$. The lattice configuration corresponding to the separatrix *ABC* has been shown in Fig. 2.

(b) *EFE type*. The orbits pass the saddle points $(\pm(\Delta_2/\sigma_2)^{1/2}, 0)$, so we have $G=\Delta_2^2/\sigma_2$. The boundary condition is $|g|=\infty$ for $x=\pm\infty$ and $|g|=$const for $x=0$. We have

$$g(x)=\frac{\alpha}{\beta}\left[\frac{\sigma_1}{\sigma_2}\right]^{1/2}\cosh(\beta x), \tag{21}$$

$$f_2^2=\frac{1}{\sigma_1+\sigma_2 g^4}\left\{\Delta_1+\Delta_2 g^2\pm\alpha\left[g^2-\frac{\sigma_1}{\sigma_2}\left[\frac{\alpha}{\beta}\right]^2\right]^{1/2}\right\}, \tag{22}$$

with $f_1=gf_2$.

(ii) At $\omega_e=[\omega_1^2+(1/2)\eta\cos(2\phi)]^{1/2}$, the first antibifurcation occurs. The fixed points contract and for

$$[\omega_1^2+\tfrac{1}{2}\eta\cos(2\phi)]^{1/2}<\omega_e<[\omega_2^2+\tfrac{1}{2}\eta\cos(2\phi)]^{1/2}, \tag{23}$$

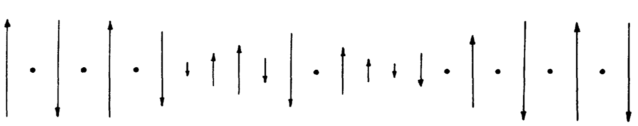

FIG. 2. The lattice pattern corresponding to the separatrix *ABC* in Fig. 1(a).

where $\Delta_1<0$ and $\Delta_2>0$, the system possesses a saddle point at (0,0) and two centers at $[\pm(\Delta_2/\sigma_2)^{1/2},0]$. The phase portrait for this case is sketched in Fig. 3(a). The separatrixes *OAB* and *OCD* (homoclinic orbits) correspond to solitons, which have been shown in Figs. 3(b) and 3(c), respectively. If the external drive, described by η, is small, then (23) becomes $\omega_1<\omega_e<\omega_2$. So the localized structures found in this case are *gap solitons.* The envelope of the light-atom vibrations have the shape of standard solitons. In the same time, the heavy-atom oscillations are also soliton but the shapes are different from the standard envelope soliton.

To find solutions corresponding to the gap solitons we note that the separatrix curves on the phase plane correspond to $G=0$. Te boundary condition for g is $|g|=$const for $x=\pm\infty$ and $|g|=\infty$ for $x=0$. Integrating (16) yields the exact solution

$$g=\pm[|\Delta_1|/\Delta_2]^{1/2}\coth y, \tag{24}$$

$$f_1=\delta_1(2\Delta_2)^{1/2}|\Delta_1|\frac{\operatorname{sech} y}{[\sigma_2\Delta_1^2+\sigma_1\Delta_2^2\tanh^4 y]^{1/2}}, \tag{25}$$

$$f_2=-\delta_1(2|\Delta_1|)^{1/2}\Delta_2\frac{\tanh y\operatorname{sech} y}{[\sigma_2\Delta_1^2+\sigma_1\Delta_2^2\tanh^4 y]^{1/2}}, \tag{26}$$

with $y=(|\Delta_1|\Delta_2)^{1/2}x$, and $\delta_1=\pm1$. $\delta_1=1$ (-1) corresponds to the *OAB* (*OCD*) orbit. The lattice configuration corresponding to the *OAB* orbit is shown in Fig. 4.

(iii) At $\omega_e=[\omega_2^2+\frac{1}{2}\eta\cos(2\phi)]^{1/2}$, the second antibifurcation occurs. The fixed points contract further and for $\omega_e>[\omega_2^2+\frac{1}{2}\eta\cos(2\phi)]^{1/2}$, where $\Delta_1<0$ and $\Delta_2<0$, the system has unique center point (0,0). So no localized structure can be found in this case.

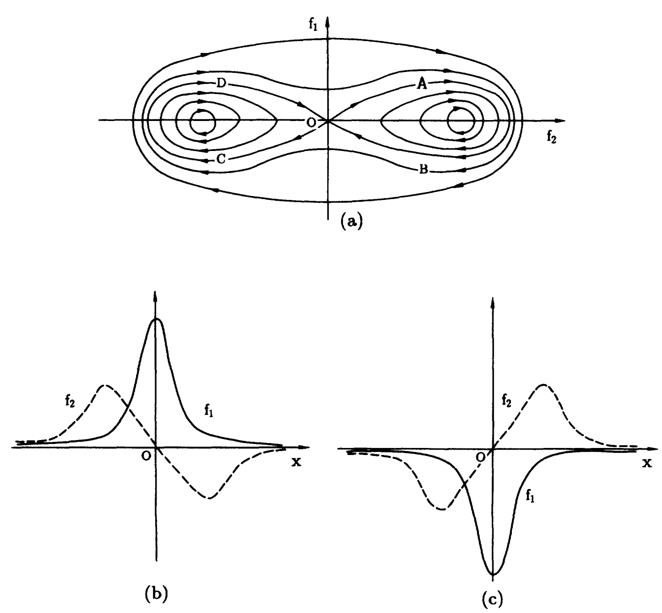

FIG. 3. (a) The phase portrait for $[\omega_1^2+\frac{1}{2}\eta\cos(2\phi)]<\omega_e<[\omega_2^2+\frac{1}{2}\eta\cos(2\phi)]$; (b) the field distribution corresponding to the homoclinic orbit *OAB*; (c) the field distribution corresponding to the homoclinic orbit *OCD*.

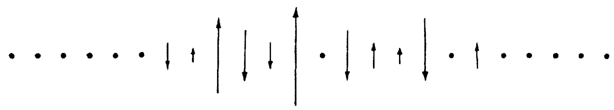

FIG. 4. The gap soliton lattice pattern corresponding to the separatrix OAB in Fig. 3(a).

Next we discuss the existence condition of the localized structures presented above. From (12) we have $\beta_1/m=\beta_2/M$. For the damping force of the pendulum we can use Stokes's formula

$$F=6\pi R\eta_v u \ , \tag{27}$$

if the velocity of the pendulum ball is not large [16]. Where R is the radius of the pendulum ball, η_v is the dynamical viscosity of air and u is the velocity of the pendulum ball. So we have $\beta_1=6\pi\eta_v R_m$ and $\beta_2=6\pi n_v R_M$, where $R_m(R_M)$ is the radius of the pendulum ball with mass $m\,(M)$. It is easy to show that in this case the condition (12) becomes

$$R_M=\left[\frac{\rho_m}{\rho_M}\right]^{1/2} R_m \ , \tag{28}$$

where $\rho_m(\rho_M)$ is the density of mass corresponding to the pendulum ball with mass $m\,(M)$. For the iron and glass ball we have $\rho_{Fe}=7.86\ \mathrm{g\,cm^{-3}}$ and ρ_{glass} (barosilicate glass)$=2.3\ \mathrm{g\,cm^{-3}}$, respectively. Thus we have $R_{glass}=1.85\ R_{Fe}$. If $R_{Fe}=1.0$ cm we have $R_{glass}=1.85$ cm, and when $R_{Fe}=0.5$ cm we have $R_{glass}=0.93$ cm. These requirements can be easily realized in experiment.

The solitons, kinks, and domain walls observed in damped and parametrically driven nonlinear monoatomic lattices clearly show that in discrete systems we may have many new localized structures resulting from nonlinearity. The experiment made by Denardo and co-workers [2,6] may be viewed as an analogy and demonstration for the nonlinear excitations in anharmonic atomic chains. Although considerable theoretical study has been made for the gap solitons in periodic media and anharmonic diatomic lattices [11–15], the experimental observation of them is absent. The results presented above suggest that the damped and parametrically driven diatomic pendulum lattice may be a good system to display the gap soliton patterns.

In conclusion, we have analytically investigated the nonlinear localized structures in a damped and parametrically driven diatomic pendulum lattice. The results show that the gap solitons may exist in this system and the exact solutions and lattice patterns for these localized structures are given. An experiment for observing the gap solitons in an anharmonic diatomic lattice is suggested.

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